Kakimizu complexes of Surfaces and 3-Manifolds

Jennifer Schultens

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Abstract

The Kakimizu complex is usually defined in the context of knots, where it is known to be quasi-Euclidean. We here generalize the definition of the Kakimizu complex to surfaces and 3-manifolds (with or without boundary). Interestingly, in the setting of surfaces, the complexes and the techniques turn out to replicate those used to study the Torelli group, *i.e.*, the "nonlinear" subgroup of the mapping class group. Our main results are that the Kakimizu complexes of a surface are contractible and that they need not be quasi-Euclidean. It follows that there exist (product) 3-manifolds whose Kakimizu complexes are not quasi-Euclidean.

The existence of Seifert's algorithm, discovered by Herbert Seifert, proves, among other things, that every knot admits a Seifert surface. *I.e.*, for every knot K, there is a compact orientable surface whose boundary is K. It is worth noting that the existence of a Seifert surface for a knot K also follows from the existence of submanifolds representing homology classes of manifolds or pairs of submanifolds, in this case the pair (K, \mathbb{S}^3) . This point of view proves useful in generalizing our understanding of Seifert surfaces to other classes of surfaces in 3-manifolds.

Adding a trivial handle to a Seifert surface produces an isotopically distinct surface. Adding additional handles produces infinitely many isotopically distinct surfaces. These are not the multitudes of surfaces of primary interest here. The multitudes of surfaces of primary interest here are, for example, the infinite collection of Seifert surfaces produced by Eisner, see [4]. Eisner realized that "spinning" a Seifert surface around the decomposing annulus of a connected sum of two non fibered knots produces homeomorphic but non isotopic Seifert surfaces. This abundance of Seifert surfaces led Kakimizu to define a complex, now named after him, whose vertices are isotopy classes of Seifert surfaces of a given knot and whose *n*-simplices are (n + 1)tuples of vertices that admit pairwise disjoint representatives.

Our understanding of the topology and geometry of the Kakimizu complex continues to evolve. Both Kakimizu's work and, independently, a result of Scharlemann and Thompson, imply that the Kakimizu complex is connected. (See [15] and [24].) Sakuma and Shackleton exhibit diameter bounds in terms of the genus of a knot. (See [23].) P. Przytycki and the author establish that the Kakimizu complex is contractible. (See [20].) Finally, Johnson, Pelayo and Wilson prove that the Kakimizu complex of a knot is quasi-Euclidean. (See [22].)

This paper grew out of a desire to study concrete examples of Kakimizu complexes of 3-manifolds other than knot complements. A natural case to consider is product manifolds, where relevant information is captured by the surface factor. The challenge lies in adapting the idea of the Kakimizu complex to a more general setting: codimension 1 submanifolds of n-manifolds.

As it turns out, in the case of 1-dimensional submanifolds of a surface, the Kakimizu complexes are related to the homology curve complexes investigated by Hatcher (see [9]), Irmer (see [13]), Bestvina-Bux-Margalit (see [2]) and Hatcher-Margalit (see [10]) discussed in Section 2. These complexes are of interest in the study of the Torelli group, which is the kernel of the action of the mapping class group of a manifold on the homology of the manifold. The Torelli group of a surface, in turn, acts on the homology curve complexes. This group action has been used to study the topology of the Torelli group of a surface, for instance by Bestvina-Bux-Margalit in their investigation of the dimension of the Torelli group (see [2]), by Irmer in "The Chillingworth class is a signed stable length" (see [12]), by Hatcher-Margalit in "Generating the Torelli group" (see [10]), and by Putman in "Small generating sets for the Torelli group" (see [21]).

Hatcher proved that the homology curve complex is contractible and computed its dimension. Irmer studied geodesics of the homology curve complex and exhibited quasi-flats. These insights guide our investigation of the Kakimizu complex of a surface. Specifically, we prove similar, and in some cases analogous, results in the setting of the Kakimizu complex of a surface. Our main results are that the Kakimizu complexes of a surface are contractible and that they need not be quasi-Euclidean.

One example stands out: The Kakimizu complex of a genus 2 surface. In [2], Bestvina-Bux-Margalit reprove a theorem of Mess, that the Torelli group of a genus 2 surface is an infinitely generated free group. They do so by showing that it acts on a tree with infinitely many edges emerging from each vertex. As it turns out, the Kakimizu complex of the genus 2 surface is also a tree with infinitely many edges emerging from each vertex. In particular, the Kakimizu complex of the genus 2 surface as a factor will thus also have some Gromov hyperbolic Kakimizu complexes. This is interesting as it shows that in addition to examples of 3-manifolds with quasi-Euclidean Kakimizu complexes, as proved by Johnson-Pelayo-Wilson, there are 3-manifolds with Gromov hyperbolic Kakimizu complexes. Kakimizu complexes exhibit more than one geometry!

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1 The Kakimizu complex of a surface

The work here follows in the footsteps of [20]. Whereas the setting for [20] is surfaces in 3-manifolds, the setting here is 1-manifolds in 2-manifolds. It is worth pointing out that although we discuss only 1-manifolds in 2-manifolds and 2-manifolds in 3-manifolds, the definitions and arguments carry over verbatim to the setting of codimension 1 submanifolds in manifolds of any dimension.

Recall that an element of a finitely generated free abelian group G is *primitive* if it is an element of a basis for G. In the following we will always assume: 1) S is a compact (possibly closed) connected oriented 2-manifold; 2) α is a primitive element of $H_1(S, \partial S, \mathbf{Z})$.

Definition 1. A Seifert curve for (S, α) is a pair (w, c), where c is a union, $c_1 \sqcup \cdots \sqcup c_n$, of pairwise disjoint oriented simple closed curves and arcs in S and w is an n-tuple of natural numbers (w^1, \ldots, w^n) such that the homology class $w^1[[c_1]] + \cdots + w^n[[c_n]]$ equals α . Moreover, we require that $S \setminus c$ is connected. We call c the underlying curve of (w, c). We will denote $w^1[[c_1]] + \cdots + w^n[[c_n]]$ by $w \circ c$.

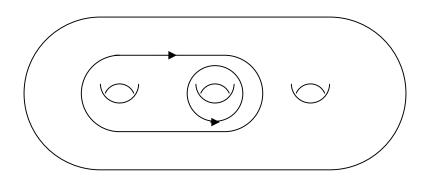


Figure 1: A Seifert curve (weights are 1)

Our definition of Seifert curve disallows null homologous subsets. Indeed, a null homologous subset would bound a component of $S \setminus c$ and would hence be separating. In fact, c contains no bounding subsets. Conversely, if $w \circ d = \alpha$ and d contains no bounding subsets, then $S \setminus d$ is connected.

Lemma 1. If (w, c) represents α , then w is determined by the underlying curve c.

Proof: Suppose that (w, c) and (w', c) represent α , where $w = (w^1, \dots, w^n)$ and $w' = ((w')^1, \dots, (w')^n)$. Then

$$w^{1}[[c_{1}]] + \dots + w^{n}[[c_{n}]] = \alpha = (w')^{1}[[c_{1}]] + \dots + (w')^{n}[[c_{n}]],$$

hence

$$(w^1 - (w')^1)[[c_1]] + \cdots (w^n - (w')^n)[[c_n]] = 0.$$

Since c has no null homologous subsets, this ensures that

$$w^{1} - (w')^{1} = 0, \dots, w^{n} - (w')^{n} = 0.$$

Thus

$$w^{1} = (w')^{1}, \cdots, w^{n} = (w')^{n}.$$

Since the underlying curve c of a Seifert curve (w, c) determines w, we will occasionally speak of a Seifert curve c, when w does not feature in our discussion.

Definition 2. Given a Seifert curve (w, c) we denote the curve obtained by replacing, for all *i*, the curve c_i with w_i parallel components of c_i , by h(w, c). This defines a function from Seifert curves to unweighted curves.

Conversely, let $d = d_1 \sqcup \cdots \sqcup d_m$ be a disjoint union of (unweighted) pairwise disjoint simple closed curves and arcs such that parallel components are oriented to be parallel oriented curves and arcs. We denote the weighted curve obtained by replacing parallel components with one weighted component whose weight is equivalent to the number of these parallel components by $h^{-1}(d)$.

Definition 3. For each pair (S, α) , the isomorphism between $H_1(S, \partial S)$ and $H^1(S)$ identifies an element a^* of $H^1(S)$ corresponding to α that lifts to a homomorphism $h_a: \pi_1(S) \to \mathbb{Z}$. We denote the covering space corresponding to $N_\alpha = kernel(h_a)$ by $(p_\alpha, \hat{S}_\alpha, S)$, or simply (p, \hat{S}, S) , and call it the infinite cyclic covering space associated with α .

We now describe the Kakimizu complex of (S, α) . As vertices we take Seifert curves (w, c) of (S, α) , considered up to isotopy of underlying curves. We write [(w, c)]. Consider a pair of vertices v, v' and representatives (w, c), (w', c'). Here $S \setminus c$ and $S \setminus c'$ are connected, hence path-connected. It follows that lifts of $S \setminus c$ and $S \setminus c'$ to the covering space associated with α are simply path components of $p^{-1}(S \setminus c)$ and $p^{-1}(S \setminus c')$. We obtain a graph $\Gamma(S, \alpha)$ by spanning an edge e = (v, v') on the vertices v, v' if and only if the representatives (w, c), (w', c') of v, v' can be chosen so that a lift of $S \setminus c$ to the covering space associated with α intersects exactly two lifts of $S \setminus c'$. (Note that in this case c and c' are necessarily disjoint.) See Figure 2.

Definition 4. Let X be a simplicial complex. If, whenever the 1-skeleton of a simplex σ is in X, the simplex σ is also in X, then X is said to be flag.

Definition 5. The Kakimizu complex of (S, α) , denoted by $Kak(S, \alpha)$ is the flag complex with $\Gamma(S, \alpha)$ as its 1-skeleton.

Remark 6. The Kakimizu complex is defined for a pair (S, α) . For simplicity we use the expression "the Kakimizu complex of a surface" in general discussions, rather than the more cumbersome "the Kakimizu complex of a pair (S, α) , where S is a surface and α is a primitive element of $H_1(S, \partial S, \mathbb{Z})$ ". Note that the Kakimizu complex of a surface is thus unique only in conjunction with a specified α .

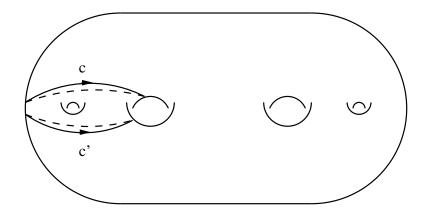


Figure 2: Two Seifert curves corresponding to vertices of distance 1 (weights are 1)

Figure 3 provides an example of a pair (w, c), (w', c') of disjoint (disconnected) Seifert curves that do not span an edge. The arc from one side of c to the other side of c intersects c' twice with the same orientation and a lift of $S \setminus c$ will hence meet at least three distinct lifts of $S \setminus c'$. For a 3-dimensional analogue of Figure 3, see [1].

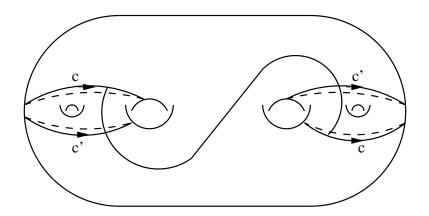


Figure 3: Two Seifert curves corresponding to vertices of distance strictly greater than 1 (weights are 1)

Example 1. The Kakimizu complexes of the disk and sphere are empty. The annulus has a non-empty but trivial Kakimizu complex $Kak(A, \alpha)$ consisting of a single vertex. Specifically, let A = annulus, and α a generator of $H_1(A, \partial A, \mathbb{Z}) = \mathbb{Z}$. Then α is represented by a spanning arc with weight 1. The spanning arc is, up to isotopy, the only possible underlying curve for a representative of α . Thus $Kak(A, \alpha)$ consists of a single vertex.

Similarly, the torus has non-empty but trivial Kakimizu complexes, each consisting of a single vertex. Specifically, let T = torus, and β a primitive element of $H_1(T, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$. Again, there is, up to isotopy, only one underlying curve for representatives of β . There are infinitely many choices for β , but in each case, $Kak(T, \beta)$ consists of a single vertex. Having understood the above Examples, we restrict our attention to the case where S is a compact orientable hyperbolic surface with geodesic boundary for the remainder of this paper.

Definition 7. Let (w, c) and (w', c') be Seifert curves. We say that (w, c) and (w', c') (or simply c and c') are almost disjoint if for all i, j the component c_i of c and the component c'_i of c' are either disjoint or coincide.

Remark 8. Let σ be a simplex in $Kak(S, \alpha)$ of dimension n. Denote the vertices of σ by v_0, \ldots, v_n and let c_0, \ldots, c_n be geodesic representatives of the underlying curves of Seifert curves for v_0, \ldots, v_n such that arc components of c_0, \ldots, c_n are perpendicular to ∂S . It is a well known fact that closed geodesics that can be isotoped to be disjoint must be disjoint or coincide. The same is true for the geodesic arcs considered here, and combinations of closed geodesics and geodesic arcs, because their doubles are closed geodesics in the double of S. Hence, for all pairs i, j, the component c_i of c and the component c'_i of c' are either disjoint or coincide.

Definition 9. Consider $Kak(S, \alpha)$. Let (p, \hat{S}, S) be the infinite cyclic cover of S associated with α . Let τ be the generator of the group of covering transformations of (p, \hat{S}, S) (which is \mathbb{Z}) corresponding to 1. Note that τ is canonical up to sign.

Let (w, c), (w', c') be Seifert curves in (S, α) . Let S_0 denote a lift of $S \setminus c$ to \hat{S} , i.e., a path component of $p^{-1}(S \setminus c)$. Set $S_i = \tau^i(S_0), c_i = closure(S_i) \cap closure(S_{i+1})$. Let S'_0 be a lift of $S \setminus c'$ to \hat{S} . Set $d_K(c, c) = 0$ and for $c \neq c'$, set $d_K(c, c')$ equal to one less than the number of translates of S_0 met by S'_0 . Let v, v' be vertices in $Kak(S, \alpha)$. Set $d_k(v, v) = 0$ and for $v \neq v'$ set $d_K(v, v')$ equal to the minimum of $d_K(c, c')$ for (w, c), (w', c') representatives of v, v'.

Definition 10. Let C, D be disjoint separating subsets of \hat{S} . We say that D lies above C if D lies in the component of $\hat{S} \setminus C$ containing $\tau(C)$. We say that D lies below C if D lies in the component of $\hat{S} \setminus C$ containing $\tau^{-1}(C)$.

Remark 11. Here $d_K(c, c')$ is finite: Indeed, $w \circ c = w' \circ c' = \alpha$ and so [(w, c)], [(w', c')]are in the kernel, N_{α} , of the cohomology class dual to α . Specifically, the cohomology class dual to α is represented by the weighted intersection pairing with (w, c) and also the weighted intersection pairing with (w', c'). Thus, let c^j be a component of c, then the value of the cohomology class dual to α evaluated at $[[c^j]]$ is given by the weighted intersection pairing of (w, c) with c^j which is 0. Likewise for other components of c and c'. Thus each component of c, c' lies in the kernel of this homomorphism and hence in N_{α} . Thus lifts of c, c' are homeomorphic to c, c', respectively, in particular, they are compact 1-manifolds. It follows that $d_K(c, c')$ is finite, whence for all vertices v, v' of $Kak(S, \alpha), d_K(v, v')$ is also finite.

It is not hard to verify, but important to note, the following theorem (see [15, Proposition 1.4]):

Proposition 1. The function d_K is a metric on the vertex set of $Kak(S, \alpha)$.

2 Relation to homology curve complexes

In [9], Hatcher introduces the cycle complex of a surface:

"By a *cycle* in a closed oriented surface S we mean a nonempty collection of finitely many disjoint oriented smooth simple closed curves. A cycle c is *reduced* if no subcycle of c is the oriented boundary of one of the complementary regions of c in S (using either orientation of the region). In particular, a reduced cycle contains no curves that bound disks in S, and no pairs of circles that are parallel but oppositely oriented.

Define the cycle complex C(S) to be the simplicial complex having as its vertices the isotopy classes of reduced cycles in S, where a set of k+1 distinct vertices spans a k-simplex if these vertices are represented by disjoint cycles c_0, \ldots, c_k that cut S into k + 1 cobordisms C_0, \ldots, C_k such that the oriented boundary of C_i is $c_{i+1} - c_i$, subscripts being taken modulo k + 1, where the orientation of C_i is induced from the given orientation of S and $-c_i$ denotes c_i with the opposite orientation. The cobordisms C_i need not be connected. The faces of a k-simplex are obtained by deleting a cycle and combining the two adjacent cobordisms into a single cobordism. One can think of a k-simplex of C(S) as a cycle of cycles. The ordering of the cycles c_0, \ldots, c_k in a k-simplex is determined up to cyclic permutation. Cycles that span a simplex represent the same element of $H_1(S)$ since they are cobordant. Thus we have a well-defined map $\pi_0: C(S) \to H_1(S)$. This has image the nonzero elements of $H_1(S)$ since on the one hand, every cycle representing a nonzero homology class contains a reduced subcycle representing the same class (subcycles of the type excluded by the definition of reduced can be discarded one by one until a reduced subcycle remains), and on the other hand, it is an elementary fact, left as an exercise, that a cycle that represents zero in $H_1(S)$ is not reduced. For a nonzero class $x \in H_1(S)$ let $C_x(S)$ be the subcomplex of C(S) spanned by vertices representing x, so $C_x(S)$ is a union of components of C(S)." See [9, Page 1].

Lemma 2. When both are defined, i.e., when S is closed, connected, of genus at least 2, and α is primitive, $Vert(Kak(S, \alpha))$ is isomorphic to a proper subset of $Vert(C_{\alpha}(S))$.

Proof: Let v be a vertex of $Kak(S, \alpha)$. If we choose a representative (w, c), then h(w, c) is a disjoint collection of (unweighted) curves and arcs. The requirement on the Seifert curve (w, c), that $S \ c$ be connected implies that the multi-curve h(w, c) is reduced and thus represents a vertex of $C_{\alpha}(S)$. Abusing notation slightly, we denote the map from $Vert(Kak(S, \alpha))$ to $Vert(C_{\alpha})$ thus obtained by h. There is an inverse, h^{-1} , defined on the image of h, hence h is injective.

It is not hard to identify reduced multi-curves that contain bounding subsets that are not the oriented boundary of a subsurface. Hence $Vert(Kak(S, \alpha))$ is a proper subset of $Vert(C_{\alpha}(S))$.

Lemma 3. Suppose that S is hyperbolic and let σ be an n-simplex in $Kak(S, \alpha)$. Denote the vertices of σ by v_0, \ldots, v_n . Then there are representatives of v_0, \ldots, v_n with underlying curves c_0, \ldots, c_n such that the following hold:

- 1. $c_i \cap c_j = \emptyset \ \forall i \neq j;$
- 2. $S \setminus (c_0 \cup \cdots \cup c_n)$ is partitioned into subsurfaces P_0, \ldots, P_n such that $\partial P_i = c_i c_{i-1}$.

Proof: Let (p, \hat{S}, S) be the covering space associated with α and let σ be a simplex in $Kak(S, \alpha)$. Let c_0, \ldots, c_n be geodesic representatives of the underlying curves of v_0, \ldots, v_n such that arc components of c_0, \ldots, c_n are perpendicular to ∂S . By Remark 8, c_i and c_j are almost disjoint $\forall i \neq j$. Consider a lift S_0 of $S \setminus c_0$ to \hat{S} . For each $j \neq 0, c_j$ lifts to a separating collection \hat{c}_j of simple closed curves and simple arcs. Moreover, since S_0 is homeomorphic to $S \setminus c_0$, the lifts \hat{c}_i, \hat{c}_j are almost disjoint as long as $i \neq j$. By reindexing c_0, \ldots, c_n if necessary and performing small isotopies that pull apart equal components, we can thus ensure that \hat{c}_i lies above \hat{c}_j for i > j.

Note that the lift of $S \setminus c_0$ is homeomorphic to $S \setminus c_0$. In particular, $c_i \cap c_j = \emptyset$ $\forall i \neq j$. Moreover, the surface with interior below \hat{c}_i and above \hat{c}_{i-1} projects to a subsurface P_i of S for i = 1, ..., n. The subsurfaces $P_1, ..., P_n$ exhibit the required properties.

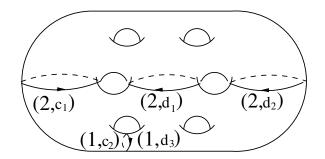


Figure 4: An edge of $Kak(S, \alpha)$ that does not map into C_{α}

Remark 12. When $w_0, \ldots, w_n = 1$, Lemma 3 ensures that $c_0 = h(w_0, c_0) = h(1, c_0)$, $\ldots, c_n = h(w_n, c_n) = h(1, c_n)$ form a cycle of cycles. In this case h extends over the simplex σ to produce a simplex $h(\sigma)$ in C_{α} . However, h does not extend over simplices in which weights are not all 1. See Figure 4.

Hatcher proves that for each $x \in H_1(S)$, $C_x(S)$ is contractible. (In particular, it is therefore connected and hence constitutes just one component of C(S).) In Section 4 we prove an analogous result for $Kak(S, \alpha)$, using a technique from the study of the Kakimizu complex of 3-manifolds.

The cyclic cycle complex and the Kakimizu complex are simplicial complexes. The complex defined by Bestvina-Bux-Margalit (see [2]) is not simplicial, but can be subdivided to obtain a simplicial complex. See the final comments in [10, Section 2]. There is a subcomplex of the cyclic cycle complex that equals this subdivision of the complex defined by Bestvina-Bux-Margalit. This is the complex of interest in the context the Torelli group. Bestvina-Bux-Margalit exploited the action of the Torelli group on this complex to compute the dimension of the Torelli group. Hatcher-Margalit used it to identify generating sets for the Torelli group.

In [13], Irmer defines the homology curve complex of a surface:

"Suppose S is a closed oriented surface. S is not required to be connected but every component is assumed to have genus $g \ge 2$.

Let α be a nontrivial element of $H_1(S, \mathbb{Z})$. The homology curve complex, $\mathcal{HC}(S, \alpha)$, is a simplicial complex whose vertex set is the set of all homotopy classes of oriented multi-curves in S in the homology class α . A set of vertices m_1, \ldots, m_k spans a simplex if there is a set of pairwise disjoint representatives of the homotopy classes.

The distance, $d_{\mathcal{H}}(v_1, v_2)$, between two vertices v_1 and v_2 is defined to be the distance in the path metric of the one-skeleton, where all edges have length one." (See [13, Page 1].)

It is not hard to see the following (*cf*, Remark 8 and Figure 3):

Lemma 4. When both are defined, i.e., when S is closed, connected, of genus at least 2 and α is primitive, $Kak(S, \alpha)$ is a subcomplex of $\mathcal{HC}(S, \alpha)$. Moreover, for vertices v, v' of $Kak(S, \alpha)$,

$$d_K(v,v') \ge d_{\mathcal{H}}(v,v')$$

Irmer shows that distance between vertices of $\mathcal{HC}(S,\alpha)$ is bounded above by a linear function on the intersection number of representatives. The same is true for vertices of the Kakimizu complex. Irmer also constructs quasi-flats in $\mathcal{HC}(S,\alpha)$. Her construction carries over to the setting of the Kakimizu complex. See Section 6.

3 The projection map, distances and geodesics

In [15], Kakimizu defined a map on the vertices of the Kakimizu complex of a knot. He used this map to prove several things, for instance that the metric, d_K , on the vertices of the Kakimizu complex equals graph distance. (Quoted and reproved here as Theorem 2.) In [20], Kakimizu's map was rebranded as a projection map.

We wish to define

$$\pi_{Vert(Kak(S,\alpha))} : Vert(Kak(S,\alpha)) \to Vert(Kak(S,\alpha))$$

on the vertex set of $Kak(S, \alpha)$. Let (p, \hat{S}, S) be the infinite cyclic covering space associated with α . Let v, v' be vertices in $Kak(S, \alpha)$ such that $v \neq v'$. Here v = [(w, c)]for some compact oriented 1-manifold c and v' = [(w', c')] for some compact oriented 1-manifold c'. We may assume, in accordance with Definition 9 and Remark 11, that (w, c) and (w', c') are chosen so that $d_K(c, c') = d_K(v, v')$. Define τ, S_i, S'_i, c_i and, by analogy, c'_i , as in Definition 9.

Instead of working only with c'_0 , we will now also work with $h(w', c'_0)$. Take $m = max\{i \mid S_{i+1} \cap S'_0 \neq \emptyset\}$. Consider a connected component C of $S_{m+1} \cap S'_0$. Its frontier consists of a subset of c'_0 and a subset of c_m . The subset of c'_0 lies above the

subset of c_m . In particular, C lies above c_m and below c'_0 , hence the orientations of the subset of c'_0 are opposite those of the subset of c_m . See Figure 5. Because the subset of c'_0 and the subset of c_m cobound C, they are homologous. It follows that the lowest components of the corresponding subset of $h(w', c'_0)$ are also homologous to the subset of c_m .

Replacing the lowest of the corresponding subsets of $h(w', c'_0)$ with the subset of c_m and isotoping this portion of c_m to lie below c_m yields a multi-curve d_1 with the following properties:

- d_1 is homologous to $h(w', c'_0)$ via a homology that descends to a homology in S (because C is homeomorphic to a subset of S);
- d_1 has lower geometric intersection number with c_m than $h(w', c'_0)$;
- d_1 lies above $h(w', c'_{-1})$ and can be isotoped to lie below and thus be disjoint from $h(w', c'_0)$, moreover its projection can be isotoped to be disjoint from h(w', c').
- For $(x_1, e_1) = h^{-1}(d_1)$, we have $x_1 \circ e_1$ homologous to $w' \circ c'_0$ via a homology that descends to a homology in S.

See Figures 5, 6.

Working with $h^{-1}(d_1), d_1$ instead of $c'_0, h(w', c'_0)$ we perform such replacements in succession to obtain a sequence of multi-curves d_1, \ldots, d_k such that the following hold:

- d_j is homologous to d_{j-1} via a homology that descends to a homology in S;
- d_j has lower geometric intersection number with c_m than d_{j-1} ;
- d_j can be isotoped to lie below $h(w', c'_0), d_1, \ldots, d_{j-1}$, moreover its projection can be isotoped to be disjoint from h(w', c').
- For $(x_j, e_j) = h^{-1}(d_j)$, we have $x_j \circ e_j$ homologues to $w' \circ c'_0$ via a homology that descends to S.
- d_k lies above $h(w', c'_{-1})$ and below c_m .

See Figures 7, 8 and 9. This proves the following:

Lemma 5. The homology class $[[p(d_k)]] = p_{\#}(x_k \circ e_k) = \alpha$.

We make two observations: 1) A result of Oertel, see [19], shows that the isotopy class of $p(e_k)$ does not depend on the choices made; 2) It is important to realize that $(x_k, p(e_k))$ may not be a Seifert curve, because $S \setminus p(e_k)$ is not necessarily connected.

If $S \setminus p(h^{-1}(e_k))$ is connected, set $p_c(c') = (x_k, p(e_k))$. Otherwise, choose a component D of $S \setminus p(e_k)$. If the frontier of D is null homologous, then remove the frontier of D from $p(e_k)$. See Figures 10, 11, 12.

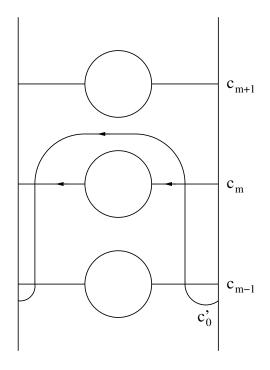


Figure 5: The setup with c_m, c'_0 (weights are 1)

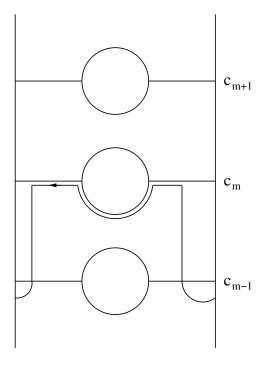


Figure 6: d_1

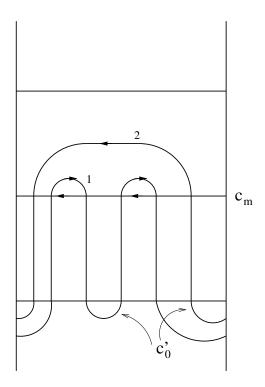


Figure 7: A different pair of weighted multi-curves

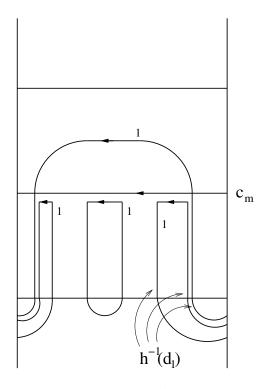


Figure 8: $h^{-1}(d_1)$

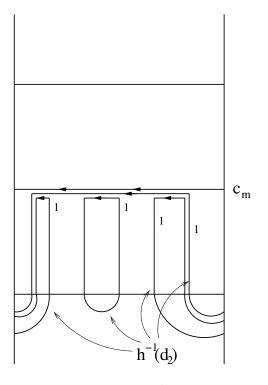


Figure 9: $h^{-1}(d_2)$

If the frontier of D is not null homologous (because the orientations do not match up) choose an arc a in its frontier with smallest weight. Denote the weight of a by w^a . We eliminate the component a of $p(e_k)$ by adding $\pm w^a$ to the weights of the other components of $p(e_k)$ in the frontier of D in such a way that the resulting weighted multi-curve still has homology α .

After a finite number of such eliminations, we obtain a weighted multi-curve that is a subset of $p(e_k)$, has homology α , and whose complement in S is connected. After reversing orientation on components with negative weights, we obtain a Seifert curve $p_c(c')$.

Lemma 6. The homology class $[[p_c(c')]] = \alpha$.

Proof: This follows from Lemma 5 and the observations above.

Definition 13. We denote the isotopy class $[p_c(c')]$ by $\pi_v(v')$.

Lemma 7. For $v \neq v'$, the following hold:

$$d_K(\pi_v(v'), v') = 1$$

and

$$d_K(\pi_v(v'), v) \le d_K(v', v) - 1$$

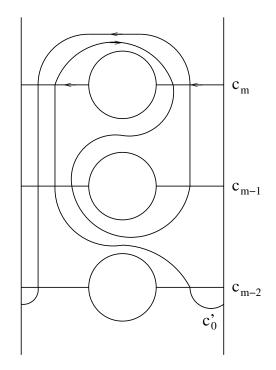


Figure 10: The setup with c_m, c'_0 (weights are 1)

It will follow from Theorem 2 below that the inequality is in fact an equality.

Proof: By construction, e_k lies strictly between c'_0 and c'_{-1} . So $\tau(e_k)$ lies strictly between c'_1 and c'_0 . Thus the lift of $S \setminus p_c(c')$ with frontier in $e_k \cup \tau(e_k)$ meets S'_0 and S'_1 and is disjoint from S'_i for $i \neq 0, 1$. It follows that the lift of $S \setminus p_c(c')$ with frontier contained in $e_k \cup \tau(e_k)$ also meets S'_0 and S'_1 and is disjoint from S'_i for $i \neq 0, 1$. It follows that the lift of $S \setminus p_c(c')$ with frontier contained in $e_k \cup \tau(e_k)$ also meets S'_0 and S'_1 and is disjoint from S'_i for $i \neq 0, 1$. Hence

$$d_K(\pi_v(v'), v') = 1.$$

In addition, suppose that $c'_0 \cap S_i \neq \emptyset$ if and only if $i \in \{n, \ldots, m+1\}$. Then $c'_1 \cap S_i \neq \emptyset$ if and only if $i \in \{n+1, \ldots, m+2\}$. Hence the lift of $S \setminus c'$ that lies strictly between c'_0 and c'_1 meets exactly S_n, \ldots, S_{m+2} .

By construction, $e_k \cap S_i$ can be non empty only if $i \in \{n, \ldots, m\}$ and thus $\tau(e_k) \cap S_i$ can be non empty only if $i \in \{n+1, \ldots, m+1\}$. Hence the lift of $S \setminus p_c(c')$ with frontier in $e_k \cup \tau(e_k)$ can meet S_i only if $i \in \{n, \ldots, m+1\}$. It follows that the lift of $S \setminus p_c(c')$ with frontier contained in $e_k \cup \tau(e_k)$ can meet S_i only if $i \in \{n, \ldots, m+1\}$. Whence

$$d_K(\pi_v(v'), v) \le m + 1 - n - 1 = d_K(v', v) - 1.$$

Definition 14. The graph distance on a complex C is a function that assigns to each pair of vertices v, v' the least possible number of edges in an edge path in C from v to v'.

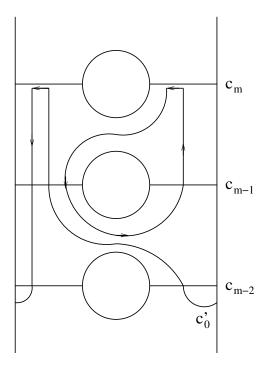


Figure 11: d_k

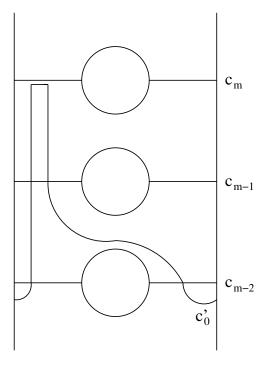


Figure 12: A subset of d_k

Theorem 2. (Kakimizu) The function d_K equals graph distance.

Proof: Denote the graph distance between v' and v by d(v', v). If $d_K(v', v) = 1$, then d(v', v) = 1 and vice versa by definition. So suppose $d_K(v', v) = m > 1$ and consider the path with vertices

$$v', \pi_v(v'), \pi_v^2(v'), \dots, \pi_v^{m-1}(v'), \pi_v^m(v') = v.$$

By Remark 7, $d_K(\pi_v(v'), v') = 1$ and $d_K(\pi_v^i(v'), \pi_v^{i-1}(v')) = 1$. Thus the existence of this path guarantees that $d(v', v) \leq m$. Hence $d(v', v) \leq d_K(v', v)$. Let $v' = v_0, v_1, \ldots, v_n = v$ be the vertices of a path realizing d(v', v). By the triangle inequality and the fact that $d(v_{i-1}, v_i) = 1 = d_K(v_{i-1}, v_i)$:

$$d_K(v',v) \le d_K(v_0,v_1) + \dots + d_K(v_{n-1},v_n) = 1 + \dots + 1 = d(v_0,v_1) + \dots + d(v_{n-1},v_n) = d(v',v)$$

The following theorem is a reinterpretation of a theorem of Scharlemann and Thompson, see [24], that was proved using different methods:

Theorem 3. The Kakimizu complex is connected.

Proof: Let v, v' be vertices in $Kak(S, \alpha)$. By Remark 11, $d_K(v, v')$ is finite. By Theorem 2, d(v, v') is finite. In particular, there is a path between v and v'. \Box

Definition 15. A geodesic between vertices v, v' in a Kakimizu complex is an edgepath that realizes d(v, v').

Theorem 4. The path with vertices $v', \pi_v(v'), \pi_v^2(v'), \ldots, \pi_v(v')^{m-1}, \pi_v^m(v') = v$ is a geodesic.

Proof: This follows from Theorem 2 because the path

$$v', \pi_v(v'), \pi_v^2(v'), \dots, \pi_v(v')^{m-1}, \pi_v^m(v') = v$$

realizes d(v', v).

Remark 16. Theorem 4 tells us that geodesics in the Kakimizu complex joining two given vertices are, at least theoretically, constructible.

Note that, typically, $\pi_v(v') \neq \pi_{v'}(v)$. See Figure 13 for a step in the construction of $\pi_{v'}(v)$.

 \square

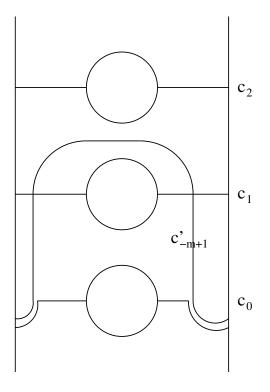


Figure 13: $u(c'_{-m+1}, c_0)$

4 Contractibility

The proof of contractibility presented here is a streamlined version of the proof given in the 3-dimensional case in [20]. Those familiar with Hatcher's work in [9], will note certain similarities with his first proof of contractibility of $C_{\alpha}(S)$ in the case that α is primitive.

Lemma 8. Suppose that v, v^1, v^2 are vertices in $Kak(S, \alpha)$. Then there are representatives c, c^1, c^2 with $v = [(w, c)], v^1 = [(w^1, c^1)], and v^2 = [(w^2, c^2)]$ that realize $d_K(v, v^1), d_K(v, v^2), and d_K(v^1, v^2).$

Proof: Let c, c^1, c^2 be geodesic representatives of the underlying curves of representatives of v, v^1, v^2 such that arc components of c, c^1, c^2 are perpendicular to ∂S . Lifts of c, c^1 , and c^2 to (p, \hat{S}, S) , the infinite cyclic covering of S associated with α , are also geodesics. Points of intersection lift to points of intersection. Geodesics that intersect can't be isotoped to be disjoint. Hence c, c^1, c^2 , with appropriate weights, realize $d_K(v, v^1), d_K(v, v^2)$, and $d_K(v^1, v^2)$.

Lemma 9. Suppose that v, v^1, v^2 are vertices in $Kak(S, \alpha)$ such that $d_K(v, v^i) > 1$ and $d_K(v^1, v^2) = 1$. Then $d_K(\pi_v(v^1), \pi_v(v^2)) \leq 1$.

Proof: In the case that, say, $v^1 = v$, note that $d_K(v^1, v^2) = 1$ means that $d_K(v, v^2) = 1$. Thus $\pi_v(v^1) = \pi_v(v) = v$ and $\pi_v(v^2) = v$. Thus $d_K(\pi_v(v^1), \pi_v(v^2)) = 0$. In the case that, say, $d_K(v^1, v) = 1$ and $v^2 \neq v$, note that $\pi_v(v^1) = v$ and

$$d_K(v, v^2) \le d_K(v, v^1) + d_K(v^1, v^2) = 1 + 1$$

thus

$$d_K(v, \pi_v(v^2) \le 1,$$

by Lemma 7, and $d_K(\pi_v(v^1), \pi_v(v^2)) \leq 1$. Hence we will assume, for the rest of this proof, that $d_K(v, v^i) > 1$.

By Lemma 8, there are representatives $(w, c), (w^1, c^1)$, and (w^2, c^2) of v, v^1 and v^2 that realize $d_K(v, v^1), d_K(v, v^2)$, and $d_K(v^1, v^2)$. Let (p, \hat{S}, S) be the infinite cyclic cover of S associated with α . Define $\tau, S_i, S_i^1, S_i^2, c_i, c_i^1, c_i^2$ as in Definition 9 but with a caveat: Label S_i^1, S_i^2 so that S_0^1, S_0^2 meet S_1 and meet S_j only if $j \leq 1$.

Since $d_K(c^1, c^2) = d_K(v^1, v^2)$, c^1 and c^2 must be disjoint. Since c_0^1 is separating, c_0^2 lies either above or below c_0^1 . Without loss of generality, we will assume that c_0^2 lies above c_0^1 (and below $\tau(c_0^1)$). See Figures 14 and 15. Note that $h(w^2, c_0^2)$ also lies above $h(w^1, c_0^1)$. Proceeding as in the discussion preceding Lemma 6, construct e_k^1 whose projection contains $p_c(c^1)$ and then e_l^2 whose projection contains $p_c(c^2)$, noting that this construction can be undertaken so that e_l^2 lies above e_k^1 (and below $\tau(e_k^1)$).

Consider the lift of $S \setminus p_c(c^2)$ with frontier in $e_l^2 \cup \tau(e_l^2)$. This lift of $S \setminus p_c(c^2)$ meets at most the two lifts of $S \setminus p_c(c^1)$ whose frontiers lie in $e_k^1 \cup \tau(e_k^1)$ and $\tau(e_k^1) \cup \tau^2(e_k^1)$. Whence

$$d_K(\pi_v(v^1), \pi_v(v^2)) \le 1.$$

Lemma 10. If $d_K(v^1, v^2) = m$, then $d_K(\pi_v(v^1), \pi_v(v^2)) \le m$.

Proof: Let $v^1 = v_0, v_1, \ldots, v_{m-1}, v_m = v^2$ be the vertices of a path from v^1 to v^2 that realizes $d_K(v^1, v^2)$. By Lemma 9, $d_K(\pi_v(v_i), \pi_v(v_{i+1})) \leq d_K(v_i, v_{i+1}) = 1$ for $i = 0, \ldots, m-1$. Hence

$$d_{K}(\pi_{v}(v^{1}),\pi_{v}(v^{2})) \leq d_{K}(\pi_{v}(v_{0}),\pi_{v}(v_{1})) + \dots + d_{K}(\pi_{v}(v_{m-1}),\pi_{v}(v_{m})) \leq d_{K}(v_{0},v_{1}) + \dots + d_{K}(v_{m-1},v_{m}) \leq m.$$

Theorem 5. The Kakimizu complex of a surface is contractible.

Proof: Let $Kak(S, \alpha)$ be a Kakimizu complex of a surface. It is well known (see [8, Exercise 11, page 358]) that it suffices to show that every finite subcomplex of $Kak(S, \alpha)$ is contained in a contractible subcomplex of $Kak(S, \alpha)$. Let \mathcal{C} be a finite subcomplex of $Kak(S, \alpha)$. Choose a vertex v in \mathcal{C} and denote by \mathcal{C}' the smallest flag complex containing every geodesic of the form given in Theorem 4 for v' a vertex in \mathcal{C} . Since \mathcal{C} is finite, it follows that \mathcal{C}' is finite.

Define $c: Vert(\mathcal{C}') \to Vert(\mathcal{C}')$ on vertices by $c(v') = \pi_v(v')$. By Lemma 9, this map extends to edges. Since \mathcal{C}' is flag, the map extends to simplices and thus to all of \mathcal{C}' . By Lemma 10 this map is continuous. It is not hard to see that c is homotopic to the identity map. In particular, c is a contraction map. (Specifically, c^d , where dis the diameter of \mathcal{C}' , has the set $\{v\}$ as its image.)

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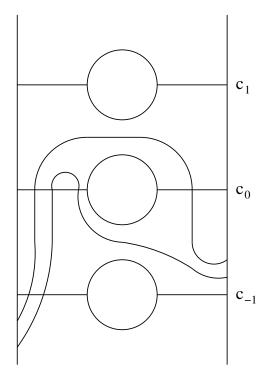


Figure 14: c_0^1 and c_0^2

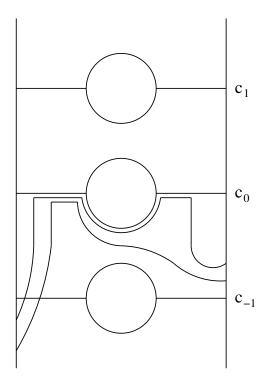


Figure 15: d_k^1 and d_l^2

5 Dimension

In [9], Hatcher proves that the dimension of $C_{\alpha}(S)$ is 2g(S) - 3, where g(S) is the genus of the closed oriented surface S. An analogous argument derives the same result in the context of $Kak(S, \alpha)$.

Lemma 11. Let S be a closed connected orientable surface with $genus(S) \ge 2$ and α a primitive class in $H_1(S, \partial S)$. The dimension of $Kak(S, \alpha)$ is $-\chi(S) - 1 = 2genus(S) - 3$.

Proof: It is not hard to build a simplex of $Kak(S, \alpha)$ of dimension 2genus(S) - 3. See for example Figure 16, where 0, 1, 2, and 3 are multi-curves (each of weight 1) representing the vertices of a simplex. Thus the dimension of $Kak(S, \alpha)$ is greater than or equal to 2genus(S) - 3.

Conversely, let σ be a simplex of maximal dimension in $Kak(S, \alpha)$. Label the vertices of σ by v_0, \ldots, v_n and let c_0, \ldots, c_n be geodesic representatives of the underlying curves of representatives of v_0, \ldots, v_n . By Lemma 3, $S \setminus (c_0 \cup \cdots \cup c_n)$ consists of subsurfaces P_0, \ldots, P_n with frontiers $c_0 - c_n, c_1 - c_0, \ldots, c_n - c_{n-1}$. Since c_i and c_{i-1} are not isotopic, no P_i can consist of annuli. In addition, no P_i can be a sphere, hence each must have negative Euler characteristic. Thus the number of P_i 's is at most $-\chi(S)$. *I.e.*,

$$n \le -\chi(S) = 2genus(S) - 2.$$

In other words, the dimension of σ and hence the dimension of $Kak(S, \alpha)$ is less than or equal to 2genus(S) - 3.

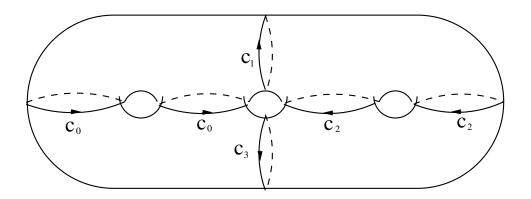


Figure 16: A simplex in a genus 3 surface

We can extend this argument to compact surfaces, by introducing the following notion of complexity:

Definition 17. Let S be a compact surface and let P be an open subset of S whose boundary consists of open subarcs of ∂S and, possibly, components of ∂S . Define

$$c(P,S) = -2\chi(P) + number of open subarcs in \partial P$$

The following lemma is immediate:

Lemma 12. Let \mathcal{C} be a union of simple closed curves and simple arcs in S. Then

$$c(S,S) = c(S \setminus \mathcal{C}, S)$$

Theorem 6. Let S be a compact connected orientable surface with $\chi(S) \leq -1$ and α a primitive class in $H_1(S, \partial S)$. The dimension of $Kak(S, \alpha)$ is $-2\chi(S) - 1 = 4genus(S) + 2b - 5$, where b is the number of boundary components of S.

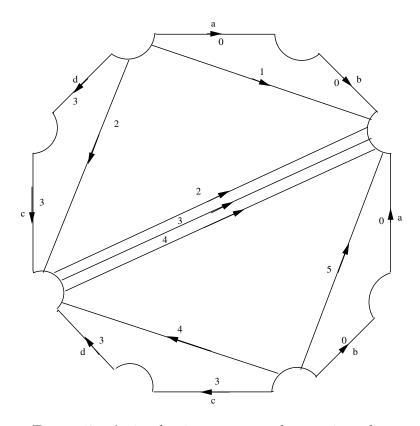


Figure 17: A simplex in a punctured genus 2 surface

Proof: To build a simplex of $Kak(S, \alpha)$ of dimension 4genus(S) + 2b - 5, see for example Figure 17, where 0, 1, 2, 3, 4 and 5 are multi-curves (each of weight 1) representing the vertices of a simplex. Thus the dimension of $Kak(S, \alpha)$ is greater than or equal to 4g(S) + 2b - 5.

Conversely, let σ be a simplex of maximal dimension in $Kak(S, \alpha)$. Label the vertices of σ by v_0, \ldots, v_n and let c_0, \ldots, c_n be geodesic representatives of the underlying curves of representatives of v_0, \ldots, v_n such that arc components of c_0, \ldots, c_n are perpendicular to ∂S . By Lemma 3, $S \setminus (c_0 \cup \cdots \cup c_n)$ consists of subsurfaces P_0, \ldots, P_n with frontiers containing $c_0 - c_n, c_1 - c_0, \ldots, c_n - c_{n-1}$. Since c_i and c_{i-1} are not isotopic, P_i can't consist of annuli or disks with exactly two open subarcs of ∂S in their boundary. In addition, no P_i can be a sphere or a disk with exactly one open

subarc of ∂S in its boundary, hence each must have positive complexity. Thus the number of P_i 's is at most $c(S, S \setminus (c_0 \cup \cdots \cup c_n))$. *I.e.*,

$$n \le c(S, S \setminus (c_0 \cup \cdots \cup c_n)) = c(S, S) = -2\chi(S).$$

In other words, the dimension of σ and hence the dimension of $Kak(S, \alpha)$ is less than or equal to $-2\chi(S) - 1 = 4genus(S) + 2b - 5$.

6 Quasi-flats

In this section we explore an idea of Irmer. See [13, Section 7].

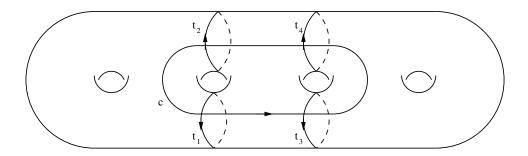


Figure 18: Building a quasi-flat by Dehn twists

Consider Figure 18. Denote the surface depicted by S and the homology class of c by α . The curves t_1 and t_2 are homologous as are t_3 and t_4 . Denote by v the vertex (1, c) of $Kak(S, \alpha)$, by v_1 the vertex corresponding to the result, d_1 , obtained from c by Dehn twisting n times around t_1 and -n times around t_2 , and by v_2 the vertex corresponding to the result, d_2 , obtained from c by Dehn twisting n times around t_4 . Then d_1, d_2 are homologous to c, so we obtain three vertices v, v_1, v_2 in $Kak(S, \alpha)$ (all weights are 1). Note the following:

$$d(v, v_i) = d_K(v, v_i) = n$$

 $d(v_1, v_2) = d_K(v_1, v_2) = n$

For i = 1, 2, we consider the geodesics g_i with vertices $v_i, \pi_v(v_i), \ldots, \pi_v^n(v_i) = v$. In addition, consider the geodesic g_3 with vertices $v_2, \pi_{v_1}(v_2), \ldots, \pi_{v_1}^n(v_2) = v_1$ and note that $\pi_{v_1}^i(v_2)$ is represented by a curve obtained from c by Dehn twisting i times around $t_1, -i$ times around $t_2, n-i$ times around t_3 and -(n-i) times around t_4 .

Definition 18. Let (X, d) be a metric space. A triangle is a 6-tuple $(v^1, v^2, v^3, g^1, g^2, g^3)$, where v^1, v^2, v^3 are vertices and the edges g^1, g^2, g^3 satisfy the following: g^1 is a distance minimizing path between v^1 and v^2 , g^2 is a distance minimizing path between v^3 and v^1 .

A triangle $(v^1, v^2, v^3, g^1, g^2, g^3)$ is δ -thin if each g^i lies in a δ -neighborhood of the other two edges. A metric space (X, d) is δ -hyperbolic if every triangle in (X, d) is δ -thin. It is hyperbolic if there is a $\delta > 0$ such that (X, d) is δ -hyperbolic.

For *n* even, the midpoint, m_1 , of the geodesic g_1 is the vertex corresponding to the result, d'_1 , obtained from *c* by Dehn twisting $\frac{n}{2}$ times around t_1 and $-\frac{n}{2}$ times around t_2 . Likewise, the midpoint, m_2 , of the geodesic g_2 is the vertex corresponding to the result, d'_2 , obtained from *c* by Dehn twisting $\frac{n}{2}$ times around t_3 and $-\frac{n}{2}$ times around t_4 . The midpoint, m_3 , of g_3 is represented by a curve obtained from *c* by Dehn twisting $\frac{n}{2}$ times around t_1 and around t_3 and $-\frac{n}{2}$ times around t_4 .

Lemma 13. Let S be the closed oriented surface of genus 4. Then $Kak(S, \alpha)$ is not hyperbolic.

Proof: For S the closed genus 4 surface, the triangle $(v, v_1, v_2, g_1, g_2, g_3)$ described depends on n, so we will denote it by T_n . In T_n we have the following:

$$d(v, m_3) = d_K(v, m_3) = n$$
$$d(v_1, m_2) = d_K(v_1, m_2) = n$$
$$d(v_2, m_1) = d_K(v_2, m_1) = n$$

In particular, g_3 is contained in a δ -neighborhood of the two geodesics g_1, g_2 only if n is less than δ . Thus the triangle T_n in $Kak(S, \alpha)$ is not δ -thin for $n \geq \delta$. It follows that $Kak(S, \alpha)$ is not hyperbolic.

Definition 19. Let (X, d) be a metric space. A quasi-flat in (X, d) is a quasi-isometry from \mathbb{R}^n to (X, d), for $n \ge 2$.

Note the following:

$$d(m_1, m_2) = d_K(m_1, m_2) = \frac{n}{2}$$
$$d(m_1, m_3) = d_K(m_1, m_3) = \frac{n}{2}$$
$$d(m_2, m_3) = d_K(m_2, m_3) = \frac{n}{2}$$

Thus the triangle T_n scales like a Euclidean triangle. It is not too hard to see that a triangle with this property can be used to construct a quasi-isometry between \mathbb{R}^2 and an infinite union of such triangles lying in $Kak(S, \alpha)$. Thus $Kak(S, \alpha)$ contains quasi-flats. It is also not hard to adapt this construction to show that, for S an oriented surface, $Kak(S, \alpha)$ is not hyperbolic and contains quasi-flats if the genus of S is greater than or equal to 4, or the genus of S is greater than or equal to 2 and $\chi(S) \leq -6$.

7 Genus 2

We consider the example of a closed orientable surface S of genus 2. A non trivial primitive homology class α can always be represented by a non separating simple closed curve with weight 1. Moreover, a Seifert curve in a closed orientable surface of genus 2, since its underlying curve is non separating, can have at most two components. Figure 19 depicts multi-curves c and $d_1 \cup d_2$ such that $[[c]] = [[d_1]] + [[d_2]]$.

We refer to a Seifert curve with one component as type 1 and a Seifert curve with two components as type 2. Since α is primitive, a Seifert curve of type 1 must have weight 1. It follows that distinct Seifert curves of type 1 must intersect. A Seifert curve of type 1 can be disjoint from a Seifert curve of type 2, see Figure 19 and distinct Seifert curves of type 2 can be disjoint, see Figure 20.

Let c be the underlying curve of a Seifert curve of type 1 and $d = d_1 \cup d_2$ the underlying curve of a Seifert curve of type 2 that are disjoint. Then the three disjoint simple closed curves $c \cup d$ cut S into pairs of pants. Any Seifert curve that is disjoint from $c \cup d$ must have underlying curve parallel to either c or d. Note that, since the weight of c is 1, the weights for d_1, d_2 must also be 1.

Consider the link of [(1, c)] in $Kak(S, \alpha)$. It consists of equivalence classes of Seifert curves of type 2. The Seifert curves of type 2 have underlying curves that are pairs of curves lying in $S \setminus c$, aren't parallel to c, and are separating in $S \setminus c$ but not in S. There are infinitely many such pairs of curves. More specifically, $S \setminus c$ is a twice punctured torus, so the curves are parallel curves that separate the two punctures and can be parametrized by \mathbf{Q} . Distinct such curves can't be isotoped to be disjoint and hence correspond to distance two vertices of $Kak(S, \alpha)$. This confirms that $Kak(S, \alpha)$ has dimension 1 = (2)(2) - 3 near [(1, c)], as prescribed by Theorem 11.

For $d = d_1 \cup d_2$ as in Figure 19 or 20, we consider $S \setminus (d_1 \cup d_2)$, a sphere with four punctures. The link of $[(w_1, w_2, d_1 \cup d_2)]$ contains isotopy classes of Seifert curves of type 1. These are essential curves that are separating in $S \setminus (d_1 \cup d_2)$ but not in S and that partition the punctures of $S \setminus (d_1 \cup d_2)$ appropriately. There are infinitely many such curves. They too can be parametrized by \mathbf{Q} . Note that distinct Seifert curves of type 1 can't be isotoped to be disjoint and hence correspond to vertices of $Kak(S, \alpha)$ of distance two or more.

In addition, the link of $[(w_1, w_2, d_1 \cup d_2)]$ contains vertices $[(u_1, u_2, e_1 \cup e_2)]$ such that one component of $e_1 \cup e_2$, say e_1 , is parallel to a component of $d_1 \cup d_2$, say d_1 and $S \setminus (d_1 \cup d_2 \cup e_1 \cup e_2)$ consists of two pairs of pants and one annulus. Seifert curves of this type can also be parametrized by Q, since e_2 is a curve in a twice punctured torus that partitions the punctures appropriately and e_1 is parallel to d_1 . Note that, since the weights of d_1, d_2 are w_1, w_2 , we must have

$$\begin{array}{rcrcrcr} w_1 &=& u_1 &\pm& u_2 \\ w_2 &=& u_2 \end{array}$$

In summary, $Kak(S, \alpha)$ is a tree each of whose vertices has a countably infinite discrete (*i.e.*, 0-dimensional) link.

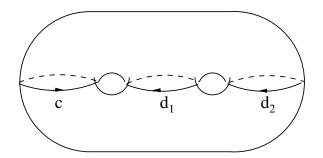


Figure 19: Underlying curves c and $d_1 \cup d_2$ in a genus 2 surface (all weights are 1)

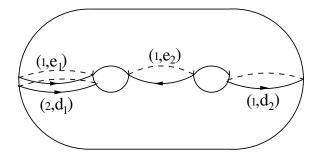


Figure 20: Seifert curves (2, 1, d) and (1, 1, e)

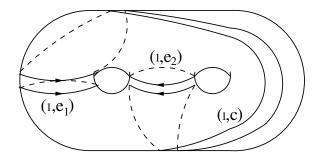


Figure 21: Seifert curves (1, 1, e) and (1, c)

Recall that Johnson, Pelayo and Wilson showed that the Kakimizu complex of a knot in the 3-sphere is quasi-Euclidean. The Kakimizu complex of the genus 2 surface is an infinite graph, thus Gromov hyperbolic. In particular, it is not quasi-Euclidean.

8 3-manifolds

The definitions given for Seifert curve, infinite cyclic cover, Kakimizu complex and so forth carry over to codimension 1 submanifolds in manifolds of any dimension. In particular, they carry over to Seifert surfaces and Kakimizu complexes in the context of compact (possibly closed) 3-manifolds. One need merely replace ones by twos and twos by threes. Instead of Seifert curves, one considers Seifert surfaces. Seifert surfaces are weighted essential surfaces that represent a given relative second homology class and have connected complement. This ties into and generalizes some of the work in [20].

Let S be a compact oriented surface. Take $M = S \times I$. Incompressible surfaces in a product manifold are either horizontal or vertical. Vertical surfaces have the form $c \times I$, where c is a multi-curve in S. It follows that $Kak(M, [[c \times I]]) = Kak(S, [[c]])$, where $[[\cdot]]$ denotes the homology class of \cdot .

Theorem 7. There exist 3-manifolds with Gromov hyperbolic Kakimizu complex.

Proof: Let S be the closed oriented surface of genus 2, α a primitive homology class in $H_1(S)$ and c a compact 1-manifold representating α . Then $Kak(S, \alpha)$ is the graph discussed in Section 7. In particular, $Kak(S, \alpha)$ is quasi-hyperbolic. Take $M = S \times I$. Then $Kak(M, [[c \times I]]) = Kak(S, \alpha)$ is also quasi-hyperbolic. \Box

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Department of Mathematics 1 Shields Avenue University of California, Davis Davis, CA 95616 USA